香港中文大學（深圳）
The Chinese University of Hong Kong, Shenzhen

SCHOOL OF DATA SCIENCE
數據科學學院

CSC 6011 - Theory of Computation

Lecture 16 - Savitch's Theorem, PSPACE-Completeness

Time: Mon & Wed, 4:00 - 5:20 PM

Location: Teaching A, 107

Instructor: Tao LIN (林涛)

TA: Yefan GAO (高叶繁)

Fall 2026

Lecture 16 - Savitch's Theorem, PSPACE-Completeness

Last time:

- Space complexity
- $\mathrm{SPACE}(f(n))$, $\mathrm{NSPACE}(f(n))$, PSPACE, NPSPACE
- Relationship with TIME classes
- TQBF, $\mathrm{LADDER}_{\mathrm{DFA}} \in$ PSPACE

Today: (Sipser §8.1-8.3)

- Review $\mathrm{LADDER}_{\mathrm{DFA}} \in$ PSPACE
- Savitch's Theorem: $\mathrm{NSPACE}(f(n)) \subseteq \mathrm{SPACE}(f^2(n))$
- PSPACE-completeness
- TQBF is PSPACE-complete

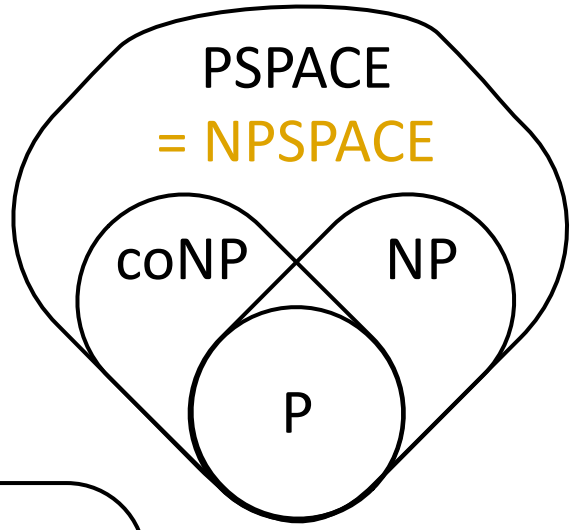
Review: SPACE Complexity

Defn: Let $f: \mathbb{N} \to \mathbb{N}$ where $f(n) \geq n$.

- A TM M runs in space $f(n)$ if M always halts and uses at most $f(n)$ tape cells on all inputs of length n.
- An NTM M runs in space $f(n)$ if all branches halt and each branch uses at most $f(n)$ tape cells on all inputs of length n.

- $\mathrm{SPACE}(f(n)) = \{B \mid \text{some 1-tape TM decides } B \text{ in space } O(f(n))\}$
- $\mathrm{NSPACE}(f(n)) = \{B \mid \text{some 1-tape NTM decides } B \text{ in space } O(f(n))\}$

- PSPACE $= \bigcup_k \mathrm{SPACE}(n^k)$ "polynomial space"
- NPSPACE $= \bigcup_k \mathrm{NSPACE}(n^k)$ "nondeterministic polynomial space"



Today: PSPACE = NPSPACE
(corollary of Savitch's Theorem)

Or possibly: P = NP = coNP = PSPACE

Review: $\text{LADDER}_{\text{DFA}} \in \text{PSPACE}$

Recall: $\text{LADDER}_{\text{DFA}} \in \text{SPACE}(n^2)$

Proof: Write $u \xrightarrow{b} v$ if there's a ladder from u to v of length $\leq b$.

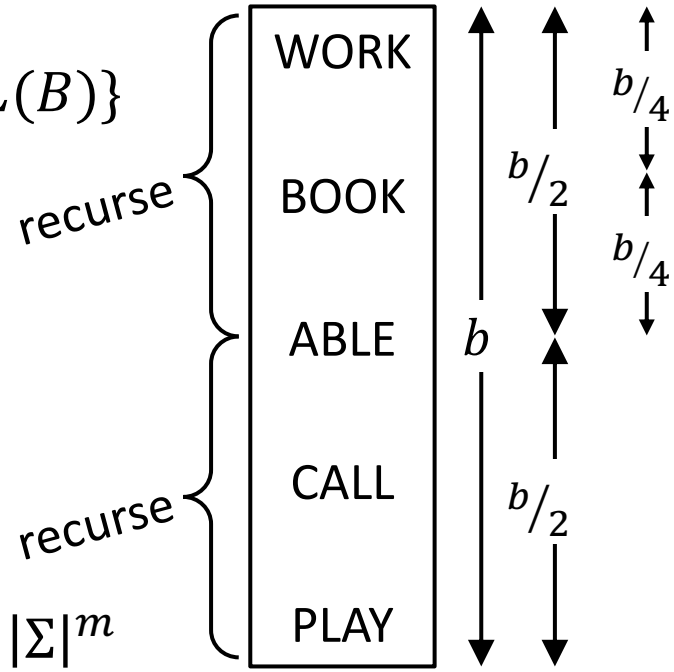
A bisection-based recursive algorithm:

$\text{BOUNDED-LADDER}_{\text{DFA}} = \{\langle B, u, v, b\rangle \mid B \text{ a DFA and } u \xrightarrow{b} v \text{ by a ladder in } L(B)\}$

B-L = "On input $\langle B, u, v, b\rangle$ Let $m = |u| = |v|$.

1. For $b = 1$, accept if $u, v \in L(B)$ and differ in ≤ 1 place, else reject.
2. For $b > 1$, repeat for each $w \in L(B)$ of length $|u|$
3. Recursively test $u \xrightarrow{b/2} w$ and $w \xrightarrow{b/2} v$ [division rounds up]
4. Accept if both accept.
5. Reject [if all fail]."

Test $\langle B, u, v\rangle \in \text{LADDER}_{\text{DFA}}$ with B-L procedure on input $\langle B, u, v, t\rangle$ for $t = |\Sigma|^m$

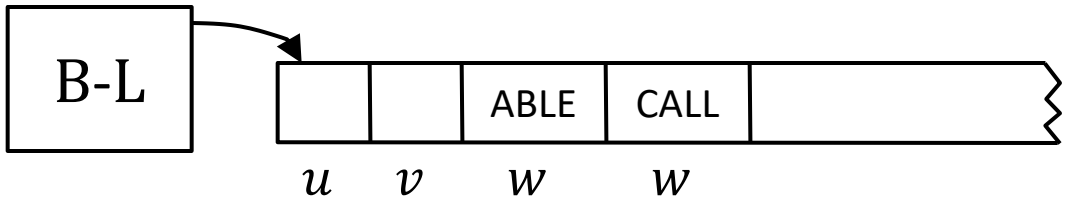


Space analysis:

Each recursive level uses space $O(n)$ (to record w).

Recursion depth is $\log_2 t = O(m) = O(n)$.

Total space used is $O(n^2)$.



PSPACE = NPSPACE

Savitch's Theorem: For $f(n) \geq n$, $\mathrm{NSPACE}\big(f(n)\big) \subseteq \mathrm{SPACE}\big(f^2(n)\big)$

Proof: Convert NTM N to equivalent TM M, only squaring the space used.

For configurations c_i and c_j of N, write $c_i \xrightarrow{b} c_j$ if can get from c_i to c_j in $\leq b$ steps.

Give recursive algorithm to test $c_i \xrightarrow{b} c_j$:

M = "On input c_i, c_j, b [goal is to check $c_i \xrightarrow{b} c_j$]

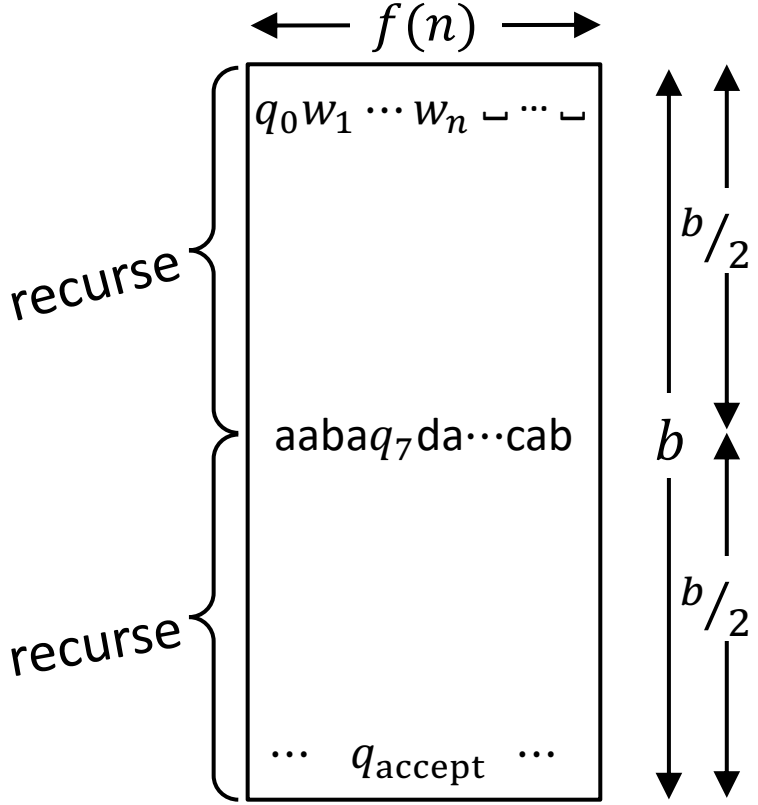
1. If $b = 1$, check directly by using N's program and answer accordingly.
2. If $b > 1$, repeat for all configurations c_{m} that use $f(n)$ space.
3. Recursively test $c_i \xrightarrow{b/2} c_{\mathrm{m}}$ and $c_{\mathrm{m}} \xrightarrow{b/2} c_j$
4. If both are true, accept. If not, continue.
5. Reject if haven't yet accepted."

Test if N accepts w by testing $c_{\mathrm{start}} \xrightarrow{t} c_{\mathrm{accept}}$

where t = number of configurations = $|Q| \times f(n) \times d^{f(n)}$

Each recursion level stores 1 configuration = $O\big(f(n)\big)$ space.

Number of levels = $\log_2 t = O\big(f(n)\big)$. Total $O\big(f^2(n)\big)$ space.



PSPACE-completeness

Defn: B is PSPACE-complete if

1) $B \in$ PSPACE

2) For all $A \in$ PSPACE, $A \leq_m^P B$

If B is PSPACE-complete and $B \in$ P then P = PSPACE.

Mini-Quiz 16.1

Knowing that TQBF is PSPACE-complete, what can we conclude if TQBF $\in$ NP? Check all that apply.

(a) P = PSPACE

(b) NP = PSPACE

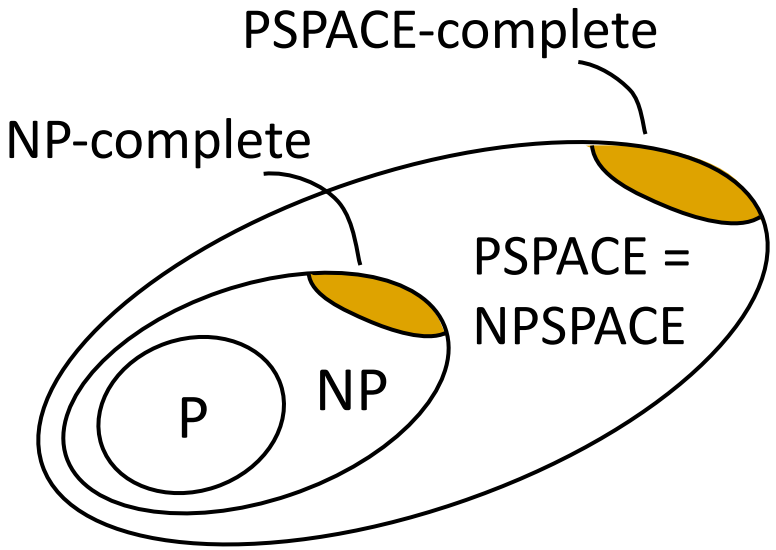
(c) P = NP

(d) NP = coNP

Why $\leq_m^P$ and not $\leq_m^{\text{PSPACE}}$ when defining PSPACE-completeness?

- Reductions should be "weaker" than the class. Otherwise all problems in the class would be reducible to each other, and then all problems in the class would be complete.

Theorem: TQBF is PSPACE-complete



Complete problems are the "hardest" in their associated class.

TQBF is PSPACE-complete

Recall: $\text{TQBF} = \{\langle\phi\rangle | \ \phi \text{ is a QBF that is TRUE}\}$

Examples: $\phi_1 = \forall x \, \exists y \, [(x \vee y) \wedge (\overline{x} \vee \overline{y})] \in \text{TQBF}$ [TRUE]

$\phi_2 = \exists y \, \forall x \, [(x \vee y) \wedge (\overline{x} \vee \overline{y})] \notin \text{TQBF}$ [FALSE]

Theorem: TQBF is PSPACE-complete

Proof: 1) $\text{TQBF} \in$ PSPACE ✓

2) For all $A \in$ PSPACE, $A \leq_m^P \text{TQBF}$

Let $A \in$ PSPACE be decided by TM M in space n^k.

Give a polynomial-time reduction f mapping A to TQBF.

$f: \Sigma^* \rightarrow$ QBFs

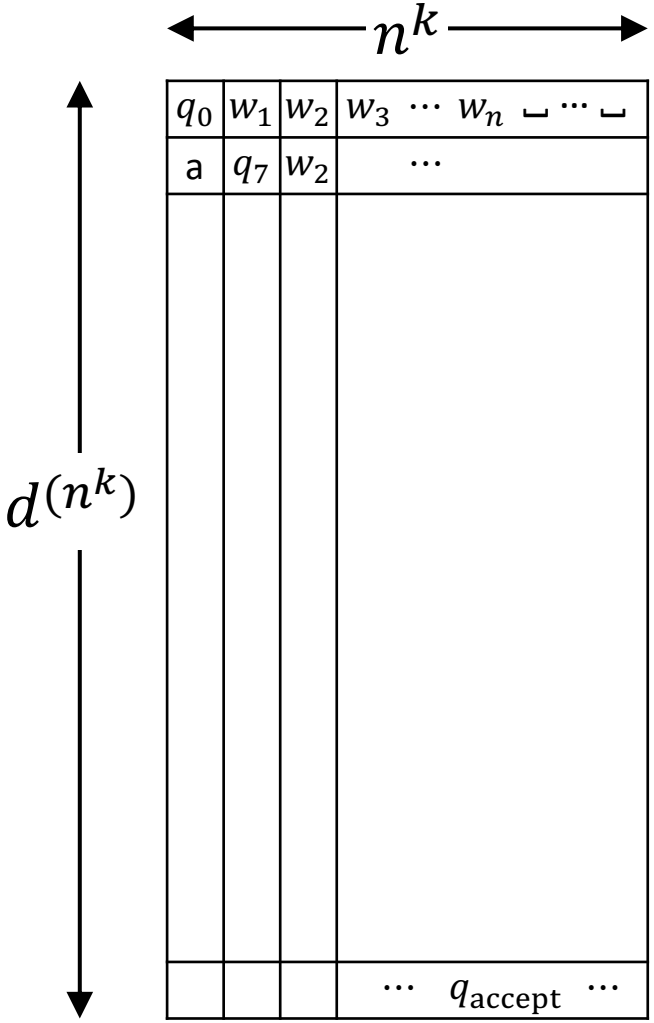
$f(w) = \langle\phi_{M,w}\rangle$

$w \in A$ iff $\phi_{M,w}$ is TRUE

Plan: Design $\phi_{M,w}$ to "say" M accepts w. $\phi_{M,w}$ simulates M on w.

Constructing $\phi_{M,w}$: 1st try

Table of computation history



Recall: A table of computation history where M accepts w. Rows are configurations.

Because M runs in space n^k, the table has:

- n^k columns (max length of a configuration)
- $d^{(n^k)}$ rows (max number of steps)

Constructing $\phi_{M,w}$. Try Cook-Levin method.

Then $\phi_{M,w}$ will be as big as table.

But that is exponential: $n^k \times d^{(n^k)}$.

Too big! ☹

Constructing $\phi_{M,w}$: 2nd try

***Q:** Why shouldn't we be surprised that this construction fails?*

For configurations c_i and c_j, construct $\phi_{c_i, c_j, b}$ which "says" $c_i \xrightarrow{b} c_j$ recursively. *We haven't used ∀*

$$\phi_{c_i, c_j, b} = \underbrace{\exists c_{\text{m}}}\left[\phi_{c_i, c_{\text{m}}, b/2} \wedge \phi_{c_{\text{m}}, c_j, b/2} \right]$$

$\exists x_1, x_2, \cdots, x_l$ satisfying conditions like φ_{cell} in Cook-Levin

$$\exists c'_{\text{m}}\left[\phi_{c_i, c'_{\text{m}}, b/4} \wedge \phi_{c'_{\text{m}}, c_{\text{m}}, b/4}\right] \qquad \exists c''_{\text{m}}\left[\phi_{c_{\text{m}}, c''_{\text{m}}, b/4} \wedge \phi_{c''_{\text{m}}, c_j, b/4}\right]$$

$$\exists c_{\text{m}}[\phi_{,,b/8} \wedge \phi_{,,b/8}]$$

$\phi_{,,1}$ is similar to φ_{move} in Cook-Levin

$$\phi_{M,w} = \phi_{c_{\text{start}}, c_{\text{accept}}, t} \quad \text{with } t = d^{(n^k)}$$

Not poly-time: Because we introduce an $\exists c_{\text{m}}$ for each middle configuration, the size of $\phi_{M,w}$ is $O\left(d^{(n^k)}\right)$ ☹

Constructing $\phi_{M,w}$: 3rd try

$\forall(x \in S)\ [\ \psi\ ]$
is equivalent to
$\forall x\ [(x \in S) \rightarrow \psi]$

$$\phi_{c_i,\, c_j,\, b} = \exists c_{\mathrm{m}} \left[\underbrace{\phi_{c_i,\, c_{\mathrm{m}},\, b/2} \ \wedge \ \phi_{c_{\mathrm{m}},\, c_j,\, b/2}} \right]$$

"There exists a middle config such that for both branches in layer 1:

$$\forall (c_g, c_h) \in \{ (c_i, c_{\mathrm{m}}), (c_{\mathrm{m}}, c_j) \} \left[\phi_{c_g,\, c_h,\, b/2} \right]$$

there exists a middle config such that for both branches in layer 2:

$$\exists c'_{\mathrm{m}} \forall (c'_g, c'_h) \in \{ (c_g, c'_{\mathrm{m}}), (c'_{\mathrm{m}}, c_h) \} \left[\phi_{c'_g, c'_h,\, b/4} \right]$$

there exists a middle config such that for both branches in layer 3:

$$\exists c''_{\mathrm{m}} \forall (c''_g, c''_h) \in \{ (c'_g, c''_{\mathrm{m}}), (c''_{\mathrm{m}}, c'_h) \} \left[\phi_{c''_g, c''_h,\, b/8} \right]$$

⋮ ⋮

we have a valid one-step move. "

$\phi_{,,1}$ is similar to φ_{move} in Cook-Levin

Constructing $\phi_{M,w}$: 3rd try

Suppose $(u_0, v_0) = (c_i, c_j)$ and $b = 2^L$

$$\phi_{c_i,c_j,b} = \exists m_1 \, \forall (u_1, v_1) \in \{(u_0, m_1), (m_1, v_0)\}$$

$$\exists m_2 \, \forall (u_2, v_2) \in \{(u_1, m_2), (m_2, v_1)\}$$

$$\vdots$$

$$\exists m_L \, \forall (u_L, v_L) \in \{(u_{L-1}, m_L), (m_L, v_{L-1})\} \, [\phi_{u_L,v_L,1}]$$

Mini-Quiz 16.2

Would this construction still work if M were nondeterministic?

(a) Yes.

(b) No.

Alternative proof for Savitch's theorem

$$\phi_{M,w} = \phi_{c_{\text{start}}, c_{\text{accept}}, t}$$
with $t = d^{(n^k)}$

Size analysis:

Each recursive level adds $O(n^k)$ to the QBF.

Number of levels $L = \log_2 d^{(n^k)} = O(n^k)$.

→ Size is $O(n^k \times n^k) = O(n^{2k})$ ☺

Quick review of today

1. $\mathrm{LADDER}_{\mathrm{DFA}} \in$ PSPACE
2. Savitch's Theorem: $\mathrm{NSPACE}\big(f(n)\big) \subseteq \mathrm{SPACE}\big(f^2(n)\big)$

   So, NPSPACE = PSPACE
3. TQBF is PSPACE-complete

Midterm Policy

1. Bring your ID card (school ID, national ID, or passport).
2. Time: Nov 9 (Monday), 16:00 – 17:20 (75~80 min, considering possible delay at the beginning).
3. Location: same classroom (TA 107)
4. Open book: you can bring any paper materials
5. Digital devices are not allowed, including phones, digital papers, smart watches, smart glasses, etc.
6. Do not open the exam paper until instructed to do so.

These slides are adapted from Michael Sipser, *18.404J Theory of Computation*, Fall 2020. Massachusetts Institute of Technology:

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